SPECTRUM:

$$\langle \tilde{x}(\omega)\, \tilde{x}^*(\omega') \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle x(t)\, x(t+T) \rangle\, e^{-i(\omega-\omega')t}\, e^{-i\omega T}\, dt\, dT$$

USING:
$$\boxed{2\pi\delta(\omega-\omega') = \int_{-\infty}^{\infty} e^{-i(\omega-\omega')t}\, dt}$$

This yields:

$$\langle \tilde{x}(\omega)\, \tilde{x}^*(\omega') \rangle = 2\pi\delta(\omega-\omega') \int_{-\infty}^{\infty} dT\, \underbrace{\langle x(t)\, x(t+T) \rangle}_{C_x(T)}\, e^{-i\omega T}$$

The spectrum is thus given by, i.e. the

~~[illegible]~~ WIENER KHINCHIN THEOREM:

$$\boxed{\begin{aligned} \langle \tilde{x}(\omega)\, \tilde{x}^*(\omega') \rangle &= 2\pi\, \delta(\omega-\omega') \cdot S(\omega) \\ S(\omega) &= \int_{-\infty}^{\infty} dT\, \langle x(t)\, x(t+T) \rangle\, e^{-i\omega T} \end{aligned}}$$

Note that in the above calculation we allowed infinite integration time. Considering finite integration we obtain: (T)

$$\langle \tilde{x}_T(\omega)\, \tilde{x}_T^*(\omega') \rangle = 2\pi\, \delta(\omega-\omega')\, S(\omega) \quad \wedge \quad \tilde{x}_T(\omega): \text{ Gated F.T.}$$

$$\boxed{S(\omega) = \lim_{T\to\infty} \frac{1}{T} |x_T(\omega)|^2} \qquad \boxed{x_T(\omega) \equiv \int_0^T dt\, e^{-i\omega t}\, x(t)}$$

Note: one can also define the Fourier transform differently for this case: e.g.
$$\tilde{x}'_T(\omega) = \underbrace{\frac{1}{\sqrt{T}}}_{\text{normalization.}} \int_0^T x(t)\, e^{-i\omega t}\, dt$$

$$\Rightarrow \quad S(\omega) \mathrel{\hat{=}} \langle \tilde{x}'_T(\omega)\, \tilde{x}'^*_T(\omega) \rangle$$

Note:
Mathematical Wiener-Khinchin theorem.

$$\boxed{\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega T}\, G(T) \\ G(T) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S(\omega)\, e^{+i\omega T} \end{aligned}}$$

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Note on SPECTRAL DENSITIES AND

We consider Fourier transforms of stochastic MARKOV processes $x(t)$. Consider the ENSEMBLE AVERAGE $\langle \ldots \rangle$

$$\langle x(t)\, x(t+\tau)\rangle \equiv G(\tau) \; (\hat{=}\; G_{xx}(\tau))$$

For an ERGODIC PROCESS. next we can define the FOURIER TRANSFORM OF A STOCHASTIC PROCESS.

TWO COMMENTS:

(i) The definition of the F.T. differs. Here we consider (RISKEN)

$$\boxed{\begin{aligned} \tilde{x}(\omega) &= \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \quad (*) \\ x(t) &= \frac{1}{2\pi}\int \tilde{x}(\omega)\, e^{+i\omega t}\, d\omega \end{aligned}}$$

Note: same def as MATHEMATICA.

Note that the $1/2\pi$ factor can also be moved to $(*)$ as done in GARDINER (ZOLLER).

(ii) The mathematical definition does not capture measurement PROCESS. CONSIDER GATED FOURIER TRANSFORM:

$$\boxed{\tilde{x}_T(\omega) = \int_0^T x(t)\, e^{-i\omega t}\, dt}$$

T: measurement time

The Definition of the Spectrum $S_{xx}(\omega)$

First properties of the Fourier Transform of stochastic variables:

$$\langle \tilde{x}(\omega)\rangle = \int dt\, e^{-i\omega t}\langle x(t)\rangle = 0$$

Since $x(t)$ is a REAL VALUED PROCESS: $x(t) = x(t)^*$

Thus: $\boxed{\tilde{x}(\omega) = \tilde{x}^*(-\omega)}$

[Since $(x(t))^* = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\tilde{x}(\omega)\, e^{i\omega t}\, d\omega)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^*(\omega)\, e^{-i\omega t}\, d\omega$

$= \frac{1}{2\pi}\int_{\infty}^{-\infty}(-d\omega)\, e^{+i\omega t}\,\tilde{x}^*(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{x}^*[-\omega]\, e^{+i\omega t}$]

The spectrum is defined via:

$$\langle \tilde{x}(\omega)\, \tilde{x}^*(\omega')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle x(t)\, x(\underbrace{t'}_{\hat{=}\, t+\tau})\rangle\, e^{-i\omega t}\, e^{+i\omega(t+\tau)}\, dt\, d\tau$$

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References:
- Martin Ch.3
- Gillespie, Introduction Mathematics of Brownian Motion and Johnson Noise
- Callen & Welton "irreversibility and generalized noise)

SPECTRAL DENSITIES

An advantage of the Markovian description of noise is the insight into the fluctuation spectrum.

$G(\tau) = C(\tau) = \langle x(t)\, x(t+\tau)\rangle$ autocorrelation function

Note: for a real function $G(\tau) \in \mathbb{R}$ it follows that $S(\omega) = S(-\omega)$. ie the spectrum is symmetric. This is not the case in quantum mechanics i.e for $x(t) \to \hat{x}(t)$ (operator obeying commutator relations)

WIENER KHINCHIN THEOREM:

$$\boxed{\begin{aligned} S_{xx}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, C(\tau)\, d\tau \\ G(\tau) &= \int_{-\infty}^{\infty} e^{+i\omega\tau} S_{xx}(\omega)\, d\omega = 2\int_{0}^{\infty} S_{xx}(\omega)\cos(\omega\tau)\, d\omega \end{aligned}}$$

Note $\frac{d\omega}{2\pi} = d\nu$

Note: The units of S_{xx} are $\left[\frac{m^2}{Hz}\right]$ for frequency fluctuations: $\left[\frac{Hz^2}{Hz}\right]$ $S_{\omega\omega}$

Examples for spectral density

Johnson Noise (RL circuit)

$V(t) = \sqrt{2k_B T R}\;\Gamma(t)$ Voltage fluctuations (example of the F.D. Theorem)

or:

$$\langle V(t) V(t+\tau)\rangle = 2k_B T R\, \delta(\tau)$$

The spectrum $S_{VV}(\omega)$ of voltage fluctuations is white (flat, i.e frequency independent)

$$S_{VV}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} \underbrace{\langle V(t)V(t+\tau)\rangle}_{\delta(\tau)2k_BTR} = 2k_B T\cdot R \quad \text{(two sided spectrum)}.$$

i.e $\boxed{\overset{\text{(single)}}{S_{VV}(\nu)} = 4 k_B T R}$ → infinite intensity process

Note that $\langle V(t)^2\rangle \to \infty$ since $\langle V(t)^2\rangle = \int_{-\infty}^{\infty} S_{VV}(\omega)\frac{d\omega}{2\pi} = \int_0^{\infty} S_{VV}(\nu)\, d\nu \to \infty$

in contrast, the current fluctuations are a stable process:

$$S_{II}(\nu) = \frac{4k_BT}{R}\;\frac{1}{1+(2\pi\nu L/R)^2}$$

$\sim \frac{1}{f^2}$ noise for high frequencies.

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Note that the spectrum of Energy fluctuations again yields thermal noise as required from equipartition: $U = \frac{1}{2} L I^2$

$$\underbrace{\frac{L}{2} S_{II}}_{} = E(\nu) \Rightarrow \int_0^\infty E(\nu) d\nu = \int_0^\infty \frac{L}{2} S_{II}(\nu) d\nu = \frac{k_B T}{2}$$

(Energy in inductor) / Hz bandwidth.

Finally, another way to write the voltage fluctuations is:

$$\langle V(t) V(t') \rangle = 2 k_B T R \langle I(t) I(t') \rangle = 2 k_B T R \delta(\tau) \quad \wedge t - t' = \tau$$

Thus: (multiplying $\int_{-\infty}^{\infty} d\tau$)

$$\boxed{R = \frac{1}{2 k_B T} \int_{-\infty}^{\infty} \langle V(t) V(t') \rangle dt'}$$

"CONDUCTANCE FORMULA"

This relation can be used to construct THERMOMETERS based on NOISE (see: "Low-Temperature Thermal Noise Thermometer", Rev. Scientific Instruments 1959 F. J. Shore)

Principle: Nyquist theorem (i.e. Johnson Noise) states that voltage fluctuations in $\nu \ldots \nu + d\nu$ frequency interval are

$$\boxed{\langle V(\nu)^2 \rangle = 4 k_B T \cdot R(\nu) \cdot d\nu}$$

(again: finite since $\nu \ldots \nu + d\nu$ interval is considered)

Thus: noise power is linear in temperature.

Note that in the quantum regime this formula is not valid.

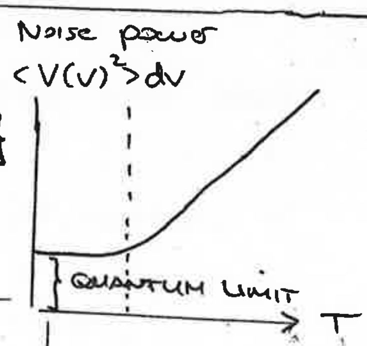
Fluctuation Dissipation Theorem by Callen & Welton "irreversibility and Generalized Noise" 1951.

$$\boxed{\langle V(\nu)^2 \rangle = 4 \cdot R(\nu) \cdot (\hbar \cdot 2\pi) \cdot \nu \Big[\big(e^{2\pi f \hbar / k_B T} - 1 \big)^{-1} + \underbrace{\tfrac{1}{2}}_{\text{ZERO POINT MOTION}} \Big] d\nu}$$

Nyquist relation with quantum mechanical correction.

$$S_{VV}(\omega) = 2\hbar\omega \coth\left(\frac{\hbar\omega}{2 k_B T} \right) \underbrace{\mathrm{Re}\{Z(\omega)\}}_{\hat{=} R}$$

$Z(\omega) = V(\omega) / I(\omega)$



Equivalently: current fluctuations

$$\boxed{\begin{aligned} S_{II}(\omega) &= \frac{2\hbar\omega}{R} \left(\coth\left(\frac{\hbar\omega}{2 k_B T} \right) \right) \\ &= \frac{4\hbar\omega}{R} \left(\frac{1}{\exp(\hbar\omega / k_B T) - 1} + \frac{1}{2} \right) \end{aligned}}$$

T_Q: $T_Q \cdot k_B \simeq \hbar\omega$

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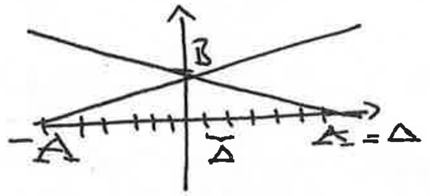
Gillespie "Twos free approach" to the Langevin Equation: A molecular impingement model of Brownian Motion.

Model Brownian motion as a 'jump process' over $(2N+1)$ States. consider the velocity v_n

stepsize $\Delta = \frac{A}{N}$

$W^{\pm}(V)$

$$V_n = \left(\frac{A}{N}\right) n \qquad n = 0, \pm 1 \dots \pm N$$



We will see that this model automatically leads to a Fluctuation–Dissipation relation, i.e a Fluctuating force and dissipation (in the limit of $N \to \infty, \Delta \to 0; A \to \infty$)

$W^{\pm}(v_n)\,dt$: probability to proceed from v_n to $v_{n\pm1}$ in time interval dt. (TRANSITION PROBABILITY)

For a symmetric random walk $\boxed{W^{-}(-v) = W^{+}(v)}$

Assuming that $\left.\begin{matrix} W_+(0) = B \\ W_+(\pm A) = 0 \end{matrix}\right\}$ thus $\begin{matrix} W_-(v) = B(1+v/A) \\ W_+(v) = B(1-v/A) \end{matrix}$

Physical interpretation: the probability of "collision" that lead to an increase in velocity decrease as v_n grows.

Hence:

$$\boxed{\begin{aligned} p(v_n, t+dt) = {} & p(v_{n-1}, t) \times W_+(v_{n-1})\,dt + P(v_{n+1}, t)\, W_-(v_{n+1})\,dt \\ & + \underbrace{p(v_n, t)\left\{1 - W_+(v_n)\,dt - W_-(v_n)\,dt\right\}}_{\text{velocity remains the same}} \end{aligned}}$$

In the limit of $dt \to 0$ rearrange by $\times 1/dt$

$$\frac{\partial p(v_n, t)}{\partial t} = p(v_{n+1}, t)\, W_-(v_{n+1}) - p(v_n, t)\, W_+(v_n) + p(v_{n-1}, t)\, W_+(v_{n-1}) - p(v_n, t)\, W_-(v_n)$$

rearranging leads to (with $W_- = B(1+v/A)$ and $W_+ = B(1-v/A)$)

$$\partial_t p(v_n, t) = \frac{B}{A}\left[v_{n+1} P(v_{n+1}, t) - v_{n-1} P(v_{n-1}, t)\right] + B\left[P(v_{n-1}, t) - 2P(v_n, t) + P(v_{n+1}, t)\right]$$

with

$\Delta = \frac{A}{N}$

Eq. with DRIFT & DIFFUSION ⇒ FPT ⇐

$$\boxed{\begin{aligned} \partial_t p(v_n, t) = {} & \frac{2B}{N}\left(\frac{v_{n+1} P(v_{n+1}, t) - v_{n-1} \cdot P(v_{n-1}, t)}{2\Delta}\right) \\ & + \frac{BA^2}{N^2}\left(\frac{P(v_{n-1}, t) - 2P(v_n, t) + P(v_{n+1}, t)}{\Delta^2}\right) \end{aligned}}$$

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This

SMOLUCHOWSKI EQUATION

$$\boxed{\frac{\partial}{\partial t}P(v,t)=C_1\frac{\partial}{\partial v}\left[v\,P(v,t)\right]+C_2\left[\frac{\partial^2}{\partial v^2}P(v,t)\right]}$$

Similar to Smoluchowski Equation with $F \propto v$ DAMPING!

SMOLUCHOWSKI WITH $F \propto v$! $\Rightarrow$ DAMPING

$$C_1 = \frac{2B}{N} \quad \wedge \quad C_2 = \frac{BA^2}{N^2}$$

limit $N \to \infty$ must approach constants which are nonzero

Hence: $A = a\sqrt{N}$ and $B = b\,N$ $\quad \therefore \left.\right] \begin{bmatrix} C_1 = 2b \\ C_2 = ba^2 \end{bmatrix}$

Solution: to smoluchowski eqs.

$$\boxed{V(t) = N\left(\underbrace{V_0 e^{-2bt}}_{\substack{\text{tends}\\\text{to zero}\\\text{(MEAN)}}}\ ;\ \underbrace{\frac{a^2}{2}\left(1-e^{-4bt}\right)}_{\text{(VARIANCE)}}\right)}$$

SOLUTION OF PROBABILITY EVOLUTION IDENTICAL TO LANGEVIN TREATMENT

Hence $V_{t\to\infty}(0, a^2/2) \overset{!}{=} V_{t\to\infty}(0;\ k_BT/M)$ Boltzmann.

Hence this implies that the RANDOM FORCE $\Rightarrow \left\{ \begin{array}{l} a = \sqrt{2k_BT/m} \\ 2b = \gamma/M \end{array}\right.$

thus $\boxed{F(t) = \sqrt{4k_BT\gamma}\ \Gamma(t)}$ RANDOM FORCE.

Model for collisions shows that they duplicate predictions of the Langevin equation.

—

Hence

$$\boxed{\partial_t P(v,t) = \frac{\gamma}{M}\frac{d}{dv}P(v,t) + \frac{1}{2}\left(2k_BT\gamma\right)\frac{\partial^2}{\partial v^2}P(v,t)}$$

FOKKER PLANCK EQS.

Fokker Planck equation with $a(x) = \gamma/M$
$b(x) = 2k_BT\gamma$

General solution:

$P(v,0) = \delta(v-v_0)$ then $P(v,t) \propto \text{Norm} \times \dfrac{1}{\sqrt{1-e^{-2C_1t}}} \times \exp\left\{\dfrac{\left(v-v_0e^{-C_1t/2}\right)^2}{2\frac{C_2}{C_1}\left(1-e^{-2C_1t}\right)}\right\}$

the stationary distribution as $t\to\infty$ becomes

$$\boxed{W(v) = e^{-(\frac{1}{2}mv^2)/(k_BT)} \cdot \sqrt{\frac{m}{2\pi k_BT}}}$$

STATIONARY DISTRIBUTION

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reference:
"Stochastic Methods"
C. Gardiner
Chapter 4

Stochastic Differential equations (Ito Calculus)

$$\frac{dx}{dt} = A(x,t) + D^{1/2}(x(t))\,\underbrace{N(t)}_{N(0,1)}\sqrt{dt}^{-1}$$

we often find definition of white Noise function $\boxed{T(t) \equiv N(t)\sqrt{dt}^{-1}}$

Note: the Langevin eqs is NOT differentiable (but continuous).

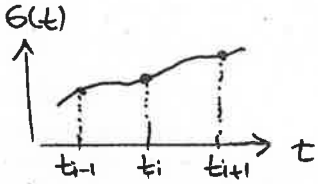
$$x(t+dt) = x(t) + A(x,t)dt + D^{1/2}(x(t),t)\cdot \underbrace{T(t)\,dt}_{\equiv dW}$$

$\boxed{dW \equiv T(t)\,dt}$ is called WIENER increment

i.e $\boxed{dx = A(x,t)\cdot dt + dW\cdot D^{1/2}(x,t)}$

Note: to use Ito calculus as ordinary computational rule there are several formulae, notably:

$$\boxed{\begin{aligned} &dW(t)^2 = dt \\ &dW(t)^{2+N} = 0 \\ * \quad &\int_{t_0}^{t} [dW(t')]^2 G(t') = \int_{t_0}^{t} dt'\,G(t') \end{aligned}}$$



* stochastic Ito integration:

$$(I)\int_{t_0}^{t} G(t')\,dW(t') = \sum_{i=1}^{n} G(t_{i-1})\,[W(t_i) - W(t_{i-1})]$$

Definition of integral.

Side comment: the stochastic integration can alternatively be defined, Stratonovich Integral (S).

ITO's formula

Consider an arbitrary function $f[x(t)]$ where $x(t)$ is determined by a S.D.E. What stochastic D.E. does $f(x(t))$ obey?

$$\begin{aligned} df(x(t)) &= f(x(t) + dx(t)) - f(x(t)) \\ &= f(x) + f'\cdot dx(t) + \tfrac{1}{2}f''(x)\,dx^2 - f(x(t)) \\ &= f'(x(t))\,\{dt\,A(x,t) + D^{1/2}(x,t)\cdot dW(t)\} + \tfrac{1}{2} f''(x,t)\,D(x,t)\cdot dW(t)^2 + O(dt^2) \end{aligned}$$

Remembering the definition $dW(t)^2 = (N(t)\sqrt{dt})^2 = dt$

we obtain: ITO's formula / LEMMA

$$\boxed{df(x(t)) = \left[A(x,t)\cdot f'(x(t)) + \tfrac{1}{2}D^{1/2}(x,t)\,f''(x,t)\right]dt + D^{1/2}(x,t)\,f'(x,t)\,dW(t)}$$

Thus: changing variables is NOT given by ordinary calculus.

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Ito's Lemma for many variabels.

$$d\vec{x} = \vec{A}(\vec{x},t)dt + B(\vec{x},t)\,d\vec{W}(t) \qquad \text{using} \qquad \begin{array}{l} dW_i\, dW_j = \delta_{ij}\, dt \\ dW_i^{N+2} = 0 \quad (N>0) \\ dt^{N+1} = 0 \end{array}$$

leads to the expression

$$\boxed{\begin{aligned} df(\vec{x}) = \Big\{ \sum_i A_i(\vec{x},t)\,\partial_i f(\vec{x}) + \frac{1}{2}\sum_{ij}\left[B(\vec{x},t)\,B^T(\vec{x},t)\right]_{ij}\partial_i\partial_j f(x)\Big\}\,dt \\ + \sum B_{ij}(\vec{x},t)\,\partial_j f(\vec{x})\, dW_j(t) \end{aligned}}$$

Examples where ITO's LEMMA plays a role.

(I) BLACK-SHOLES formula for derivative pricing (options pricing)

Nobel Prize in Economics 1997 for "A method to determine the values of derivatives" → homework exercise.

$$-rw + w_1 x r + w_2 \frac{1}{2} x^2 \sigma^2 = 0 \qquad \text{BLACK-SCHOLES DIFF. EQ.}$$

x: stock price
w: warrant price ($w(x,t)$)
t: time to maturity

r: return rate

(HOMEWORK)

(II) Complex Oscillator with a Noisy frequency.

This model plays a central role in determining the ultimate (quantum noise) limited linewidth of a laser.

$$\frac{dE}{dt} = i(\omega + \underbrace{\sqrt{2\gamma}\cdot\Gamma(t)}_{\delta\omega(t):\ \text{frequency noise}})\,E \qquad \text{or:} \quad E(t) = E_0\, e^{-i\omega t + i\Phi(t)}$$

and: $$\frac{d\Phi}{dt} = D^{1/2}\cdot\Gamma(t)$$
(WIENER PROCESS)

$$\langle \delta\omega(t)\,\delta\omega(t')\rangle = 2\gamma\,\delta(t-t')$$

(III) Geometric Brownian Motion.

$$dx = c\cdot x\, dW \qquad \text{(multiplicative noise)}$$